THE FIFTY-TWO ICOSAHEDRAL SOLUTIONS TO PAINLEVÉ VI

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ABSTRACT. The solutions of the (nonlinear) Painlevé VI differential equation having icosahedral linear monodromy group will be classified up to equivalence under Okamoto's affine F_4 Weyl group action and many properties of the solutions will be given.

There are 52 classes, the first ten of which correspond directly to the ten icosahedral entries on Schwarz's list of algebraic solutions of the hypergeometric equation. The next nine solutions are simple deformations of known $\mathrm{P_{VI}}$ solutions (and have less than five branches) and five of the larger solutions are already known, due to work of Dubrovin and Mazzocco and Kitaev.

Of the remaining 28 solutions we will find 20 explicitly using the method of [5] (via Jimbo's asymptotic formula). Amongst those constructed there is one solution that is "generic" in that its parameters lie on *none* of the affine F_4 hyperplanes, one that is equivalent to the Dubrovin–Mazzocco elliptic solution and three elliptic solutions that are related to the Valentiner three-dimensional complex reflection group, the largest having 24 branches.

CONTENTS

1. Introduction

The Painlevé VI equation ($\mathrm{P_{VI}}$) is a second order nonlinear differential equation which governs the isomonodromic deformations of linear systems of differential equations of the form

$$(1)\qquad \frac{d}{dz}-\left(\frac{A_1}{z}+\frac{A_2}{z-t}+\frac{A_3}{z-1}\right),\qquad A_i\in\mathfrak{g}:=\mathfrak{sl}_2(\mathbb{C})$$

as the second pole position t varies in $\mathbb{P}^1\setminus\{0,1,\infty\}$. (The general case—varying all four pole positions—reduces to this case using automorphisms of $\mathbb{P}^1$.)

Painlevé VI is notoriously difficult to solve explicitly, and indeed it has been proved that 'most' solutions are new transcendental functions.

Upon fixing the eigenvalues of the residues A_i (and of the residue $A_4:=-\sum_1^3 A_i$ at infinity), and identifying two systems if they are related by a constant gauge transformation,

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one obtains a moduli space

$$\boldsymbol{\mathcal{O}} := \mathcal{O}_1 \times \cdots \mathcal{O}_4 /\!\!/ G = \left\{ \ (A_1, \dots A_4) \in \mathcal{O}_1 \times \cdots \mathcal{O}_4 \ \middle| \ \sum A_i = 0 \ \right\} / G$$

of such systems, which is (complex) two-dimensional in general, where $G = \mathrm{SL}_2(\mathbb{C})$ and $\mathcal{O}_i \subset \mathfrak{sl}_2(\mathbb{C})$ is the adjoint orbit of elements having the chosen eigenvalues, assumed non-resonant here. (Traditionally one parameterises the choice of the four adjoint orbits $\mathcal{O}_i$ by four complex numbers θ_i such that A_i has eigenvalues $\pm\,\theta_i/2$.)

Geometrically (see below) each $\mathrm{P_{VI}}$ equation amounts to a (nonlinear) connection on the trivial fibre bundle

$$\mathcal{M}^* := \boldsymbol{\mathcal{O}} \times B \to B$$

where the base $B := \mathbb{P}^1 \setminus \{0, 1, \infty\}$ is the domain of t.

Thus, roughly speaking, the set

$$\mathfrak{g}^4 /\!\!/ G = \left\{ \ (A_1, \dots A_4) \in \mathfrak{g}^4 \ \middle| \ \sum A_i = 0 \ \right\} / G \tag{2}$$

of residues is foliated by a family of surfaces $\boldsymbol{\mathcal{O}}$ parameterised by $\theta_1, \dots \theta_4$ (i.e. as the θ's vary the surfaces $\boldsymbol{\mathcal{O}} \subset \mathfrak{g}^4 /\!\!/ G$ sweep out most of $\mathfrak{g}^4 /\!\!/ G$).

Now in [22] K. Okamoto has defined, again roughly speaking, a birational action of the affine Weyl group $W_a(F_4)$ on the space of systems

$$\mathfrak{g}^4 /\!\!/ G \times B.$$

(A point of this product corresponds to a choice of residues A_i and a choice of $t \in B$, and so determines a system as in (1).) The action on B is via Möbius transformations μ permuting $0, 1, \infty$ and, as we will confirm, the action on $(\theta_1, \dots, \theta_4) \in \mathbb{C}^4$ is the standard $W_a(F_4)$ action. Two key properties of Okamoto's action are:

- It maps each leaf $\boldsymbol{\mathcal{O}} \times B \subset \mathfrak{g}^4 /\!\!/ G \times B$ to another leaf, say $\boldsymbol{\mathcal{O}}' \times B$, and
- It relates the $\mathrm{P_{VI}}$ connection on $\boldsymbol{\mathcal{O}} \times B$ to the $\mathrm{P_{VI}}$ connection on $\boldsymbol{\mathcal{O}}' \times B$ (so that local $\mathrm{P_{VI}}$ solutions $s : U \to \boldsymbol{\mathcal{O}}$ for $U \subset B$ map to local solutions $s' : \mu(U) \to \boldsymbol{\mathcal{O}}'$).

The author's prior understanding of the $\mathrm{P_{VI}}$ folklore was that the classical solutions of $\mathrm{P_{VI}}$ (i.e. those which are not 'new' transcendental functions) had θ-parameters lying on one or more of the reflection hyperplanes of the $W_a(F_4)$ action (the idea being that such solutions had more symmetry).

One of the main aims of this paper is to show that this is not necessarily the case—an explicit algebraic solution will be written down with parameters on *none* of the affine F_4 hyperplanes.

To explain the strategy used let us first recall more of the geometrical path to $\mathrm{P_{VI}}$.

The monodromy representation of a linear system (1) is a point of the space

$$\boldsymbol{\mathcal{C}} = \mathrm{Hom}_{\mathcal{C}}(\pi_1(\mathbb{P}^1 \setminus \{0, t, 1, \infty\}), G)/G$$

of conjugacy classes of representations of the fundamental group of the four-punctured sphere, where the representations are restricted to take the simple loop around the ith puncture into the conjugacy class $\mathcal{C}_i := \exp(2\pi\sqrt{-1}\mathcal{O}_i) \subset G$. Upon choosing appropriate loops generating $\pi_1(\mathbb{P}^1 \setminus \{0, t, 1, \infty\})$, $\boldsymbol{\mathcal{C}}$ is identified with

$$\left\{ (M_1, \dots, M_4) \ \middle| \ M_i \in \mathcal{C}_i, \ M_4 \cdots M_1 = 1 \ \right\} / G \tag{3}$$

the multiplicative analogue of $\boldsymbol{\mathcal{O}}$, and is similarly seen to be two-dimensional in general.

As the position t of the second pole varies these surfaces $\mathcal{C}$ fit together into a fibre bundle

$$M \to B,$$

the fibre over $t \in B$ being the surface $\mathcal{C}$ associated to the four-punctured sphere $\mathbb{P}^1 \setminus \{0, t, 1, \infty\}$.

This fibre bundle $M \to B$ has a natural complete flat connection on it (in other words it is a local system of varieties): as $t \in B$ is varied slightly we can identify two nearby fibres of M by using the same loops to identify both fibres with (3) and therefore with each other.

The $\mathrm{P_{VI}}$ equation is obtained by pulling back this connection on $M \to B$ along the relative Riemann–Hilbert map

$$\nu : \mathcal{M}^* \to M$$

(taking systems to their monodromy). Choosing particular (t-dependent) coordinates on the fibres of $\mathcal{M}^*$, writing out what one gets and eliminating one coordinate, yields $\mathrm{P_{VI}}$:

$$\frac{d^2y}{dt^2} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{dy}{dt}$$
$$+\frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{(t-1)}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right)$$

where the constants $\alpha, \beta, \gamma, \delta$ are related to the θ-parameters as follows:

$$\alpha = (\theta_4 - 1)^2/2, \qquad \beta = -\theta_1^2/2, \qquad \gamma = \theta_3^2/2, \qquad \delta = (1-\theta_2^2)/2. \tag{4}$$

From this picture it is immediate that the branching of solutions to $\mathrm{P_{VI}}$ (horizontal sections of the connection on $\mathcal{M}^*$) corresponds to the monodromy of the connection on M. But the connection on M is complete and so its monodromy amounts to an action of the fundamental group $\mathcal{F}_2 = \pi_1(B)$ of the base B on a fibre $\mathcal{C}$. This action of $\mathcal{F}_2$ (the free group on two generators) can be written down explicitly in terms of the standard Hurwitz braid group action.

The simplest solutions of $\mathrm{P_{VI}}$ should be those with only a finite number of branches. For example if we take a linear system (1) which has a basis of algebraic solutions then the corresponding $\mathrm{P_{VI}}$ solution (controlling the isomonodromic deformations of (1)) will be finite branching. One way to see this is to recall that (1) will have a basis of algebraic solutions if and only if its monodromy group $\langle M_1, M_2, M_3\rangle \subset G$ is finite, i.e. the M_i are generators of a finite subgroup Γ of $G = \mathrm{SL}_2(\mathbb{C})$. Now the triple M_1, M_2, M_3 represents a point of a surface $\mathcal{C}$ and the $\mathcal{F}_2$ action on $\mathcal{C}$ (the monodromy of the $\mathrm{P_{VI}}$ solution) acts within the set of triples of generators of Γ. Thus the number of branches of the $\mathrm{P_{VI}}$ solution is bounded, e.g. by $|\Gamma|^3$. The idea of looking for solutions of $\mathrm{P_{VI}}$ starting from a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$ goes back at least to Hitchin [12] (see also [13]).

In this paper we will examine the set of $\mathrm{P_{VI}}$ solutions which arise in this way upon taking Γ to be the binary icosahedral group.

Theorem A. (Icosahedral Classification) *Upto equivalence under Okamoto's $W_a(F_4)$ action, there are precisely 52 solutions to P_{VI} having icosahedral linear monodromy group. The possible genera are $0, 1, 2, 3, 7$ and the largest solution has 72 branches.*

More details about the solutions can be found in Table 1. Examining the parameters of the solutions we find exactly one such solution turns out to have generic parameters,

corresponding to choosing one of the four generators M_1, M_2, M_3, M_4 to be in (a lift to $\mathrm{SL}_2(\mathbb{C})$ of) each of the four nontrivial conjugacy classes of A_5, the icosahedral rotation group. (The reader may like to confirm that there are such four-tuples of elements of A_5 having product the identity.) This leads to:

Theorem B. (Generic Solution) *There is an algebraic solution to the sixth Painlevé equation whose parameters lie on none of the reflecting hyperplanes of Okamoto's affine F_4 (or D_4) action.*

Proof. Set $(\theta_1, \theta_2, \theta_3, \theta_4) = (2/5, 1/2, 1/3, 4/5)$ and consider the rational functions

$$y = -\frac{9s(s^2+1)(3s-4)(15s^4-5s^3+3s^2-3s+2)}{(2s-1)^2(9s^2+4)(9s^2+3s+10)}, \quad t = \frac{27s^5(s^2+1)^2(3s-4)^3}{4(2s-1)^3(9s^2+4)^2} \qquad \square$$

The main technical tool used in the construction of this solution is the precise formula of M. Jimbo (see [16] and [5] Theorem 4) for the leading term in the asymptotic expansion at zero of generic $\mathrm{P_{VI}}$ solutions. In brief, using the $\mathrm{P_{VI}}$ equation these leading terms determine the Puiseux expansions of each branch of the solution at zero and, taking sufficiently many terms, these determine the solution completely since it is algebraic.

One of the basic facts the author came to appreciate during the construction of this generic solution is that even though two solutions may be equivalent by Okamoto transformations, the size of the polynomial defining them may well vary dramatically; one should try to choose the parameters for which they become as simple as possible (which is still something of an art). For example the first solution curve $F(y,t) = 0$ found, defining an equivalent generic solution, took a page to write down and involved about a hundred twenty-digit integers. A more perspicacious choice of parameters reduced the size of the polynomial F and enabled the above parameterisation to be computed.

This led to the question of whether there is a better choice of equivalent parameters for the elliptic icosahedral solution of Dubrovin and Mazzocco [10], for which the solution curve $F(y,t)$ took about ten pages to write down (in the preprint version of op. cit. on the math archive), and for which a parameterisation was not possible to compute.

Theorem C. *The Dubrovin–Mazzocco elliptic solution is equivalent to the solution with parameters $\theta_i = 1/3$, $i = 1,2,3,4$ given by the functions y, t on the elliptic curve*

$$u^2 = s\,(8\,s^2 - 11\,s + 8)$$

where

$$y = \frac{1}{2} - \frac{8\,s^7 - 28\,s^6 + 75\,s^5 + 31\,s^4 - 269\,s^3 + 318\,s^2 - 166\,s + 56}{18\,u\,(s-1)\,(3\,s^3 - 4\,s^2 + 4\,s + 2)}$$

$$t = \frac{1}{2} + \frac{(s+1)\,(32\,(s^8+1) - 320\,(s^7+s) + 1112\,(s^6+s^2) - 2420\,(s^5+s^3) + 3167\,s^4)}{54\,u^3\,s\,\,(s-1)}.$$

In particular this elliptic curve is birational to that defined by the 10*-page polynomial.*

We remark that this solution was constructed directly with the above parameter values, rather than by transforming the curve of Dubrovin and Mazzocco. (It is now straightforward to apply Okamoto transformations to the above parameterised solution to obtain a

parameterisation of their curve; t, u, s remain unchanged but y becomes more complicated: see Section 5.)

The author was first motivated to examine the set of icosahedral solutions to P_{VI} for the following reason. In previous articles [4, 5] the author studied an alternative realisation where P_{VI} governs the isomonodromic deformations of certain rank three Fuchsian systems having four poles on $\mathbb{P}^1$ and having rank one residues at three of the poles. In particular [5] explained how to relate these systems, and their monodromy data, to the standard SL_2 viewpoint described above. In terms of monodromy representations this yields a direct way to construct finite $\mathcal{F}_2$ orbits of $\mathrm{SL}_2(\mathbb{C})$ triples, from each triple of reflections generating a (finite) three-dimensional complex reflection group. For example the Klein solution was constructed in [5] starting from a triple of generating reflections of the smallest non-real exceptional three-dimensional complex reflection group, the Klein group. In that case the corresponding $\mathrm{SL}_2(\mathbb{C})$ triple is not equivalent to that of any finite subgroup of $\mathrm{SL}_2(\mathbb{C})$.

Somewhat disappointingly most of the other three dimensional complex reflection groups seem to lead to known Painlevé VI solutions (or simple deformations of them). However the largest exceptional three-dimensional complex reflection group, the Valentiner group (a six fold cover of A_6 of order 2160), does yield new solutions: one finds there are three inequivalent triples of generating reflections, with $\mathcal{F}_2$ orbits of sizes $15, 15, 24$ respectively, all corresponding to genus 1 solutions. However the corresponding triples in $\mathrm{SL}_2(\mathbb{C})$ all turn out to generate the binary icosahedral group. (In particular this gives an unexpected relationship between A_6 and A_5.) Thus we realised there are other interesting icosahedral solutions distinct from those previously found, and so became curious to see any others that might occur. These Valentiner solutions now appear as rows $37, 38, 46$ of Table 1, and we have managed to construct all three solutions explicitly. (Currently the 24 branch solution is the highest degree explicit algebraic solution to P_{VI}.)

Theorem D. (Valentiner Solutions) *There are three inequivalent triples of reflections generating the Valentiner complex reflection group having $\mathcal{F}_2$ orbits of sizes* $15, 15, 24$ *respectively. The corresponding P_{VI} solutions all have genus one and are equivalent to icosahedral solutions. (They will appear in Section 8.)*

Following the procedure of [5] these solutions give explicit families of rank three, four-poled Fuchsian systems having monodromy the Valentiner group, generated by reflections.

The layout of this article is as follows. Section 2 describes convenient parameters on, and enumerates, the set S of conjugacy classes of triples of generators of Γ. Section 3 then counts the set of $W_a(F_4)$ orbits of the θ-parameters that arise from S. Since Γ is finite all such θ's are real and so this amounts to reflecting the parameters into a fundamental domain (the closure of an alcove) for the real action of $W_a(F_4)$. This gives a lower bound, 52, on the number of inequivalent icosahedral solutions. Section 4 then proves 52 is also an upper bound by examining the natural action of the mapping class group (and of the centre of G^3) on triples of generators of Γ, and relating this to Okamoto's $W_a(F_4)$ action. Combined with Section 3 this gives the desired classification. Section 5 then lists many properties of the 52 solutions (and describes the relation with Schwarz's list, with Lamé equations and with the previously published icosahedral solutions of Dubrovin and Mazzocco and Kitaev). Section 6 discusses the generic icosahedral solution, then section 7 presents some other explicit icosahedral solutions, including all the outstanding genus

zero solutions as well as some others of genus one. (Remark 21, added after the rest of this paper was written, explains how the remaining solutions may be obtained.) Finally section 8 presents the Valentiner solutions, which was our starting point.[1]

Similar considerations may also be applied to the cases of the tetrahedral and octahedral groups; this has now been done (and all such P_{VI} solutions are now known explicitly). Details will appear elsewhere [3].

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2. Generating Triples

Let $G = \mathrm{SL}_2(\mathbb{C})$ and consider the binary icosahedral group $\Gamma \subset G$ of order 120. It has a center of order 2 and the quotient $\Gamma/\pm \subset \mathrm{PSL}_2(\mathbb{C}) \cong \mathrm{SO}_3(\mathbb{C})$ is the icosahedral group A_5. In terms of unit quaternions, explicit generators of Γ are [7]:

$$\Gamma = \langle\ (-1+\mathbf{i}+\mathbf{j}+\mathbf{k})/2,\ (\mathbf{i}+\sigma\mathbf{j}+\tau\mathbf{k})/2\ \rangle$$

where $\sigma = (\sqrt{5}-1)/2, \tau = (\sqrt{5}+1)/2$ and $\mathbf{i} = \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right), \mathbf{j} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), \mathbf{k} = \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)$.

Our first aim is to study the set of triples of generators of Γ. Suppose we have a triple $M_1, M_2, M_3 \in \Gamma$ which generate Γ (rather than a proper subgroup). Denote this triple by $\mathbf{M}$:

$$\mathbf{M} = (M_1, M_2, M_3).$$

Then define $M_4 \in \Gamma$ by the requirement that

$$M_4 M_3 M_2 M_1 = 1 \tag{5}$$

and consider the seven-tuple of numbers

$$\mathbf{m} = \mathbf{m}(\mathbf{M}) = (m_1, m_2, m_3, m_4, m_{12}, m_{23}, m_{13})$$

where

$$m_i := \mathrm{Tr}(M_i), \qquad m_{ij} := \mathrm{Tr}(M_i M_j).$$

Lemma 1. *Two triples of generators of Γ are conjugate (in G or Γ) if and only if they have the same seven-tuple* $\mathbf{m}$.

Sketch. This follows since these traces generate the ring of invariants of the diagonal conjugation action of G on G^3, and that Γ is its own normaliser in G. □

There is in fact a formula to count the conjugacy classes of generating triples:

Lemma 2 ([11])**.** *There are* 26688 *conjugacy classes of triples of generators of* Γ.

[1]To aid the reader who is interested in examining the solutions of this paper (for example to draw the corresponding dessins d'enfants), a Maple text file of the solutions has been included with the source file on the math arxiv (math.AG/0406281). This may be downloaded by clicking on "Other formats" and unpacked with the commands 'gunzip 0406281.tar' and 'tar -xvf 0406281.tar', at least on a Unix system.

Proof. P. Hall shows ([11] p.146) that the number of n-tuples of generators of Γ is

$$120^n - 5(24^n) - 6(20^n) - 10(12^n) + 20(6^n) + 60(4^n) - 60(2^n).$$

Setting $n = 3$ and dividing by 60 (since the center of Γ acts trivially when conjugating) gives the result. $\square$

Thus the set

$$S := \{\mathbf{m}(\mathbf{M}) \bigm| \langle M_1, M_2, M_3\rangle = \Gamma \}$$

of seven-tuples of invariants of generating triples of Γ, has cardinality 26688. Fortunately there are quite strong notions of equivalence for elements of S, which will dramatically reduce this number. In the next sections we will define and study two equivalence relations (parameter equivalence and geometric equivalence) on S, which will turn out to have the same 52 equivalence classes.

3. Parameter equivalence—affine Weyl groups

Given a seven-tuple $\mathbf{m} \in S$ we can associate four parameters

$$\theta = \theta(\mathbf{m}) := (\theta_1, \theta_2, \theta_3, \theta_4) \in \mathbb{R}^4$$

where $\theta_j \in \mathbb{R}$ is determined from $\mathbf{m}$ via:

$$m_j = 2\cos(\pi\theta_j), \qquad 0 \leqslant \theta_j \leqslant 1$$

so that the corresponding matrix $M_j \in \Gamma$ has eigenvalues $\{\exp(\pm\pi i\theta_j)\}$.

Definition 3. Two seven-tuples $\mathbf{m}, \mathbf{m}'$ are *parameter equivalent* if their parameters θ, θ' are in the same orbit of the standard action of the affine Weyl group of type F_4 on $\mathbb{R}^4$.

In order to explain this, let us briefly recall some basic facts about root systems and the corresponding affine Weyl group actions (for more details see e.g. Bourbaki [6]).

Let V be a real four-dimensional Euclidean vector space with orthonormal basis $\varepsilon_1, \dots, \varepsilon_4$. The Euclidean inner product will be denoted (u, v) and used to identify V with its dual V^*. Let $O(V)$ denote the group of linear transformations of V preserving the inner product and let $\mathrm{Aff}(V) \cong O(V) \ltimes V$ denote the group of affine Euclidean transformations of V (i.e. those of the form $v \mapsto g(v) + w$ for some $g \in O(V), w \in V$). A vector in V will be denoted $\sum \theta_i\varepsilon_i$ with $\theta_i \in \mathbb{R}$ (the indices on ε_i and θ_i will always run from 1 up to 4).

The standard F_4 root system is the following set of 48 vectors in V:

$$F_4 = \{\ \pm\varepsilon_i,\ \pm\varepsilon_i \pm \varepsilon_j (i < j),\ (\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)/2\}.$$

Each root $\alpha \in F_4$ determines a coroot $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$ as well as a hyperplane L_α in V:

$$L_\alpha := \{\ v \in V \bigm| (\alpha, v) = 0\ \}.$$

In turn α determines an orthogonal reflection s_α, the reflection in this hyperplane:

$$s_\alpha(v) = v - 2\frac{(\alpha, v)}{(\alpha, \alpha)}\alpha = v - (\alpha^\vee, v)\alpha.$$

The Weyl group $W(F_4) \subset O(V)$ is the group generated by these reflections:

$$W(F_4) = \langle\ s_\alpha \bigm| \alpha \in F_4\ \rangle$$

which is of order 1152.

Similarly the choice of a root $\alpha \in F_4$ and an integer $k \in \mathbb{Z}$ determines an affine hyperplane $L_{\alpha,k}$ in V:

$$L_{\alpha,k} := \{ \ v \in V \ \big| \ (\alpha, v) = k \ \}$$

and the reflection $s_{\alpha,k}$ in this hyperplane is an affine Euclidean transformation

$$s_{\alpha,k}(v) = s_\alpha(v) + k\alpha^\vee.$$

The affine Weyl group $W_a(F_4) \subset \mathrm{Aff}(V)$ is the group generated by these reflections:

$$W_a(F_4) = \langle \ s_{\alpha,k} \ \big| \ \alpha \in F_4, \ k \in \mathbb{Z} \ \rangle$$

which is an infinite group isomorphic to the semi-direct product of $W(F_4)$ and the coroot lattice $Q(F_4^\vee)$ (which is the lattice in V generated by the coroots $\alpha^\vee$).

Now in [22] Section 3, Okamoto defines a birational action of (a copy of) $W_a(F_4)$ on a 7-dimensional space of linear differential equations, the "total phase space" of Painlevé VI, (involving the four parameters plus the canonical coordinates p, q and the time variable t—this is essentially the space $\mathfrak{g}^4 /\!\!/ G \times B$ in the introduction, with p, q being coordinates on $\boldsymbol{\mathcal{O}}$; given an isomonodromic family of linear equations it is the function $y = q$ which solves $\mathrm{P_{VI}}$). This action descends to an action on just the space of the four parameters (denoted v_i in [22]). By relating Okamoto's four parameters to the θ-parameters used here we see that Okamoto defines an embedding $\iota : W_a(F_4) \hookrightarrow \mathrm{Aff}(V)$.

Lemma 4. *Okamoto's embedding maps his copy of $W_a(F_4)$ isomorphically onto the standard $W_a(F_4) \subset \mathrm{Aff}(V)$.*

Proof. The action of $\iota(W_a(F_4))$ is generated ([22] p.364) by the reflections in the five hyperplanes bounding the alcove:

$$v_2 > v_3 > v_4 > 0, \qquad v_1 > v_2 + v_3 + v_4, \qquad v_1 + v_2 < 1 \tag{6}$$

where $v_1 = \theta_3 - 1, v_2 = \theta_1, v_3 = \theta_2, v_4 = \theta_4 - 1$, whereas the standard $W_a(F_4)$ is generated by the reflections in the hyperplanes bounding the standard alcove:

$$\theta_2 > \theta_3 > \theta_4 > 0, \qquad \theta_1 > \theta_2 + \theta_3 + \theta_4, \qquad \theta_1 + \theta_2 < 1. \tag{7}$$

One may show $\iota(W_a(F_4)) \subset W_a(F_4)$ by finding $g \in W_a(F_4)$ mapping (6) isomorphically onto (7). (Such g may be found by applying the procedure of Proposition 6 below to an interior point of (6).) Similarly for the reverse inclusion. □

Remark 5. The reason we are being careful here and speaking of different copies of $W_a(F_4)$, is that the analogous result is not true for Okamoto's affine D_4 action. Recall from [22] that Okamoto starts by defining an action of $W_a(D_4)$ (which fixes the time variable t) and the action of $W_a(F_4)$ is obtained by adding some more generators. However when written as an action on V (our space of θ's) $W_a(D_4)$ is not embedded in $\mathrm{Aff}(V)$ as the standard $W_a(D_4)$, but rather as $W_a(D_4^-)$ where D_4^- is the set of 24 *short* roots of F_4:

$$D_4^- := \{ \ \pm\varepsilon_i, \ (\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)/2 \ \},$$

whereas the standard D_4 is the set of long roots of F_4. Moreover one then naturally has that $W_a(F_4)$ is the normaliser of $W_a(D_4^-)$ in $\mathrm{Aff}(V)$, and is an extension by S_4, the automorphisms of the extended D_4 Dynkin diagram; each $W_a(D_4^-)$ alcove is partitioned into $24 = |S_4|$ copies of the $W_a(F_4)$ alcove.

In this paper we are viewing two solutions to P_{VI} as equivalent if they are related by Okamoto's action of $W_a(F_4)$. Thus it is immediate that the 2×2 linear monodromy data of any two equivalent P_{VI} solutions will be parameter equivalent. Hence by computing the set of $W_a(F_4)$ orbits of the set $\theta(S)\subset V$ we obtain a lower bound for the set of inequivalent icosahedral P_{VI} solutions:

Proposition 6. *There are at least* 52 *inequivalent icosahedral solutions to* P_{VI}.

Proof. Direct computation—the standard procedure for computing affine Weyl group orbits is as follows. The set of affine hyperplanes partitions V into a set of disconnected pieces, the alcoves, and the affine Weyl group acts simply transitively on the set of these alcoves. Every affine Weyl group orbit intersects the closure of any alcove in exactly one point.

Thus we choose an alcove $\mathcal{A}$, so that $V/W_a(F_4)\cong\overline{\mathcal{A}}$, and for each point of $\theta(S)$ we find the corresponding point of $\overline{\mathcal{A}}$. This is done by repeatedly reflecting in the hyperplanes bounding $\mathcal{A}$ until all the inequalities determining $\overline{\mathcal{A}}$ are satisfied. (This procedure will always terminate after a finite number of steps.) Proceeding in this way we find (using Maple) that the set $\theta(S)$ leads to precisely 52 points of $\overline{\mathcal{A}}$. □

Remark 7. There is clearly lots of choice of alcove—one would perhaps eventually like to find an alcove in which the corresponding P_{VI} solutions have as simple form as possible. In the meantime we can use for example the standard alcove (7) or the alcove (6) suggested by Okamoto's work or that suggested by Noumi–Yamada's article [21]:

$$\alpha_2>0,\qquad \alpha_0>\alpha_1>\alpha_4>\alpha_3>0$$

where $\alpha_0=\theta_2,\alpha_1=\theta_4-1,\alpha_3=\theta_3,\alpha_4=\theta_1,\alpha_2=(1-\alpha_0-\alpha_1-\alpha_3-\alpha_4)/2$ (which is often convenient because the full birational action of $W_a(D_4)$ is given succinctly in [21] and its extension to $W_a(F_4)$ is written in these terms in [20] 7.14).

Next we will look for a sharp upper bound on the number of icosahedral solutions.

4. Geometric equivalence—mapping class groups

Let a_1,a_2,a_3,a_4 be four distinct points of the real two-dimensional sphere S^2 (say $a_1,\dots a_4=0,\frac{1}{2},1,\infty$) and consider the mapping class group of the sphere preserving the set of these points:

$$\mathrm{M}_{0,4}:=\pi_0(\mathrm{Diff}(S^2,\{a_1,a_2,a_3,a_4\}))$$

which is the *group* of connected components of the group of orientation preserving diffeomorphisms $f:S^2\to S^2$ such that $f(\{a_1,a_2,a_3,a_4\})=\{a_1,a_2,a_3,a_4\}$.

The following facts about $\mathrm{M}_{0,4}$ will be useful:

$\mathrm{M}_{0,4}$ is generated by elements $\omega_i,\ i=1,2,3$ where ω_i is related to the Dehn twist swapping a_i,a_{i+1} in an anti-clockwise sense. They satisfy the relations given in [2] p.164.

By mapping ω_i to the permutation $(i,i+1)\in S_4$, one obtains the exact sequence

$$1\to\mathcal{F}_2\to\mathrm{M}_{0,4}\to S_4\to 1 \tag{8}$$

where the kernel (the pure mapping class group) is isomorphic to the free group $\mathcal{F}_2$ on two letters, freely generated by ω_1^2,ω_2^2.

Write $S^* = S^2 \setminus \{a_1, a_2, a_3, a_4\}$ for the four-punctured sphere. There is a natural map $\mathrm{M}_{0,4} \to \mathrm{Out}(\pi_1(S^*))$ to the group of outer automorphisms (the group of all automorphisms modulo the inner automorphisms) of the fundamental group of the four-punctured sphere. This is defined as follows: Given $f \in \mathrm{M}_{0,4}$ one obtains an isomorphism $f_* : \pi_1(S^*) \to \pi_1(S^*)$, however the basepoint may well move, so one needs to quotient by inner automorphisms.

This map induces an action of $\mathrm{M}_{0,4}$ on the set of conjugacy classes of representations of the fundamental group of the four-punctured sphere:

$$\mathrm{M}_{0,4} \to \mathrm{Aut}(\mathrm{Hom}(\pi_1(S^*), G)/G).$$

Explicitly the generators act as follows. First choose simple positive loops γ_i around a_i generating $\pi_1(S^*)$ such that $\gamma_4 \circ \cdots \circ \gamma_1$ is contractible, and let $M_i = \rho(\gamma_i) \in G$ for any representation $\rho \in \mathrm{Hom}(\pi_1(S^*), G)$. Then ω_i fixes M_j for $j \neq i, i+1$, $(1 \leqslant j \leqslant 4)$ and

$$\omega_i(M_i,\ M_{i+1}) = (M_{i+1},\ M_{i+1}M_iM_{i+1}^{-1}).$$

In terms of the traces $m_i = \mathrm{Tr}(M_i),\ m_{ij} = \mathrm{Tr}(M_iM_j)$ generating the ring of G-invariant functions on $\mathrm{Hom}(\pi_1(S^*), G)$ one finds (as in [5] Lemma 1):

$$\begin{aligned}
\omega_1(\mathbf{m}) &= (m_2,\ m_1,\ m_3,\ m_4,\ m_{12},\ m_2m_4 + m_1m_3 - m_{13} - m_{12}m_{23},\ m_{23})\\
\omega_2(\mathbf{m}) &= (m_1,\ m_3,\ m_2,\ m_4,\ m_{13},\ m_{23},\ m_3m_4 + m_1m_2 - m_{12} - m_{23}m_{13})\\
\omega_3(\mathbf{m}) &= (m_1,\ m_2,\ m_4,\ m_3,\ m_{12},\ m_2m_4 + m_1m_3 - m_{13} - m_{12}m_{23},\ m_{23})
\end{aligned}$$

where $\mathbf{m} = (m_1,\ m_2,\ m_3,\ m_4,\ m_{12},\ m_{23},\ m_{13})$. (In computing this action we follow the conventions of [2]; in the conventions used in [5] one has $\omega_1 = \beta_2^{-1}, \omega_2 = \beta_1^{-1}$.)

Let $t_0 = 1/2$ be a basepoint of $B = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and choose loops w_1 (resp. w_2) based at t_0 encircling 0 (resp. 1) once in a positive sense. These two loops generate $\pi_1(B) \cong \mathcal{F}_2$ and there is a canonical map $\pi : \pi_1(B) \to \mathrm{M}_{0,4}$ mapping $\pi_1(B)$ isomorphically onto the pure mapping class group, $\pi(w_i) = \omega_i^2$ for $i = 1, 2$. The action of this $\mathcal{F}_2$ corresponds to the nonlinear monodromy of Painlevé VI. (To define π geometrically, first define a map $B \to ((\mathbb{P}^1)^4 \setminus \text{diagonals})/S_4$ by mapping t to the unordered set $\{0, t, 1, \infty\}$. Taking fundamental groups gives a map $\pi_1(B) \to \mathrm{SB}_4$ to the four-string spherical braid group [2] p.34. Then recall $\mathrm{M}_{0,4}$ is naturally the quotient of SB_4 by its centre [2] Theorem 4.5. The relation between the generators is described [2] p.165.)

As well as the mapping class group there is another symmetry group acting on the monodromy data that we wish to consider. Recall that there are precisely two connected Lie groups with Lie algebra $\mathfrak{sl}_2(\mathbb{C})$: the simply connected group $\mathrm{SL}_2(\mathbb{C})$ with center ± 1 and its quotient $\mathrm{SL}_2(\mathbb{C})/\pm 1 = \mathrm{PSL}_2(\mathbb{C}) \cong \mathrm{SO}_3(\mathbb{C})$. Thus any triple $\mathbf{M} = (M_1, M_2, M_3)$ projects to a triple of elements of $\mathrm{PSL}_2(\mathbb{C})$. We will say two triples $\mathbf{M}, \mathbf{M}'$ are *sign equivalent* if they project to the same triple in $\mathrm{PSL}_2(\mathbb{C})$. Said differently, let

$$\Sigma = \{\ (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) \bigm| \epsilon_i = \pm 1,\ \Pi\epsilon_i = 1\ \} \cong (\mathbb{Z}/2)^3$$

be the group of even four-tuples of signs then, since $M_4M_3M_2M_1 = 1$, we are acting on the four-tuple M_1, M_2, M_3, M_4 with Σ in the obvious way $M_i \mapsto \epsilon_i M_i$.

The mapping class group $\mathrm{M}_{0,4}$ acts on Σ via the map (8) to S_4 and the obvious action of S_4 permuting the ϵ_i, and so we may construct a larger group $\widetilde{\mathrm{M}}_{0,4}$ the semi-direct product

$$\widetilde{\mathrm{M}}_{0,4} := \mathrm{M}_{0,4} \ltimes \Sigma$$

generated by the mapping class group and the sign changes. Note that $\widetilde{\mathrm{M}}_{0,4}$ actually acts on the set of conjugacy classes of triples of generators of the binary icosahedral group, and therefore also on the set S of invariants of generating triples.

Definition 8. Two seven-tuples $\mathbf{m}, \mathbf{m}' \in S$ are *geometrically equivalent* if they are in the same orbit of the group $\widetilde{\mathrm{M}}_{0,4}$.

A key fact that we will use is:

Lemma 9. *If two solutions of P_{VI} have geometrically equivalent linear monodromy data in S, then the solutions are equivalent.*

Proof. First note that if two solutions have the same data $\mathbf{m}$ then they are related by a translation in $W_a(D_4^-)$.

Also recall that if two solutions have monodromy data related by the free subgroup $\mathcal{F}_2 \subset \mathrm{M}_{0,4}$ then they are equivalent, since the $\mathcal{F}_2$ action corresponds to the branching of a single solution.

Thus it is sufficient, for each generator of $\widetilde{\mathrm{M}}_{0,4}$, to find an Okamoto transformation inducing the same action on the monodromy data, at least up to the action of $\mathcal{F}_2$. To avoid confusion first note that there are two reasons that $W_a(F_4)$ does not in fact act on the monodromy data:

First it is straightforward to check that $W_a(F_4)$ does not even act on the local monodromy data $(m_1, \ldots m_4)$; even the subgroup $W_a(D_4^-)$ does not act here. This can be easily rectified by working with the data (θ_i, m_{ij}) instead. Thus $W_a(D_4^-)$ acts on $\{(\theta_i, m_{ij})\}$ (acting trivially on the quadratic functions m_{ij} by [14]).

Secondly one still does not get an action of $W_a(F_4)$ on $\{(\theta_i, m_{ij})\}$, since the $W_a(F_4)$ action on the systems (1) moves the pole positions, and so one obtains representations of the fundamental group of different punctured spheres. Although for each fixed t only six four-punctured spheres arise (the S_3 orbit of t) one cannot just add in S_3; one inevitably becomes involved with an infinite subgroup of $\mathrm{M}_{0,4}$. (The essential reason for this is that the sequence (8) does not split; this is implied by the non-splitting of (9) below, which follows from the Kurosh subgroup theorem.) However as we will explain below, there is a well-defined action on the set of $\mathcal{F}_2$ orbits in $\{(\theta_i, m_{ij})\}$ which is sufficient for us.

There are two steps: first the generators of $\mathrm{M}_{0,4}$ will be related, up to the $\mathcal{F}_2$ action, to certain automorphisms of $\mathbb{P}^1$. Then the action of these automorphisms on the systems (1) will be identified with Okamoto transformations. (The action of the signs will be dealt with at the end.)

Suppose $f(z)$ is a Möbius transformation such that $f(\{0, t, 1, \infty\}) = \{0, t, 1, \infty\}$ for some $t \in B := \mathbb{P}^1 \setminus \{0, 1, \infty\}$. Thus f represents an element of the mapping class group

$$\mathrm{M}_{0,4}^t := \pi_0(\mathrm{Diff}(S^2, \{0, t, 1, \infty\}))$$

of the sphere with the points $\{0, t, 1, \infty\}$ marked. For example we will be interested in the cases:

$$f_1 = \frac{t - z}{t - 1}, \qquad f_2 = \frac{z}{t}, \qquad f_3 = \frac{z}{z - 1},$$

which represent elements of the groups $\mathrm{M}_{0,4}^t$ when $t = 2, -1, 2$ respectively. Now to identify these mapping class groups $\mathrm{M}_{0,4}^t$ with a standard one, say with $\mathrm{M}_{0,4} = \mathrm{M}_{0,4}^{1/2}$ we need to choose diffeomorphisms identifying these four-pointed spheres; different choices of

diffeomorphisms correspond to conjugating by elements of $\mathcal{F}_2 \subset \mathrm{M}_{0,4}$. Thus each f_i leads to a well-defined element of $\mathrm{M}_{0,4}/\mathcal{F}_2 \cong S_4$. These elements of S_4 are just the induced permutations of $\{0, t, 1, \infty\}$, i.e.

$$f_i \text{ corresponds to the involution } (i, i+1) \in S_4.$$

Thus, up to $\mathcal{F}_2$, the action of f_i is given by the generator ω_i of $\mathrm{M}_{0,4}$, for $i = 1, 2, 3$.

Now we will identify the action, by pullback, of these Möbius transformations f_i on the systems (1) in terms of Okamoto transformations. This is straightforward since the Okamoto transformations are determined by their action on $V = \{(\theta_i)\}$. We find f_1 corresponds to the transformation x^3 of [22] p.361 (with $1/(t-1)$ corrected to $t/(t-1)$), that f_2 corresponds to $x^{313} := x^3 \circ x^1 \circ x^3$ and f_3 corresponds to $s \circ x^{313}$ where $s \in W_a(D_4^-)$ is the element acting on the θ-parameters as the permutation (14)(23). (One could check these directly or use the universality of Okamoto's action—that all such Möbius transformations will lead to Okamoto transformations.)

Finally we will obtain a generator of the sign changes (others being obtained under the action of S_4). Note that the transformation x^2 of [22] p.361 maps θ to $(\theta_4 - 1, \theta_2, \theta_3, \theta_1 + 1)$, so that $T := s \circ x^{313} \circ x^2$ maps θ to $(\theta_1 + 1, \theta_2, \theta_3, \theta_4 - 1)$. We claim T acts on the monodromy data just by negating M_1 and M_4 and fixing M_2, M_3. To see this we identify T as a composition $T = \tau \circ R$. Here R is the rational gauge transformation (elementary Schlesinger transformation [17]) increasing by 1 the first eigenvalue $\theta_1/2$ of the residue A_1 at zero, and decreasing by 1 the first eigenvalue $\theta_4/2$ of the residue A_4 at infinity. The resulting system has exactly the same monodromy data but is no longer in $\mathfrak{sl}_2$, so we apply the operation τ twisting by the flat line bundle $-\frac{dz}{2z}$ to get an $\mathfrak{sl}_2$ system with parameters $(\theta_1 + 1, \theta_2, \theta_3, \theta_4 - 1)$. The twisting gives the sign changes. □

This leads to the main result:

Theorem 1. *There are exactly* 52 *inequivalent icosahedral solutions to* P_{VI}*.*

Proof. First we compute, using Maple, the orbits of $\widetilde{\mathrm{M}}_{0,4}$ in S. We find there are exactly 52 orbits. Thus by Lemma 9 there are at most 52 inequivalent icosahedral solutions to $\mathrm{P_{VI}}$. Combining this with Proposition 6 yields result. □

Some properties of these solutions will be listed in the next section.

Remark 10. One may be interested in classifying the systems (1) having bases of algebraic solutions. The natural equivalence relation to use is geometric equivalence (of the correpsonding monodromy representations) since this preserves the (projective) monodromy group. Theorem 1 implies one will get the same classification as that appearing here (for the icosahedral representations). There is still the thorny question of determining precisely which pole positions are possible for systems having given exponents and monodromy representation. But this is determined in a straightforward way if we know the corresponding $\mathrm{P_{VI}}$ solution (namely the $\mathrm{P_{VI}}$ solution explicitly determines an isomonodromic family of systems and one examines when there are poles in the matrix entries of these systems). See also remark 18.

Remark 11.
1) In general parameter equivalence is strictly weaker than equivalence, even when restricted to algebraic solutions (cf. e.g. [12]). Adding other invariants (such as the number

of branches, the genus, the nonlinear monodromy group size) does often distinguish algebraic solutions, but not always: For example (11) and (12) below are both solutions for $(\theta_1,\theta_2,\theta_3,\theta_4)=(1,1,1,3)/6$ and have identical invariants, but are inequivalent (since they are not related by the subgroup of $W_a(F_4)$ stabilising $(1,1,1,3)/6$; each of the three reflections σ_{03},σ_{04} and $s_1s_2s_1$ of [20, 21] fix both solutions).
2) In general geometric equivalence is strictly stronger than equivalence; one needs to add the action of the rest of $W_a(D_4^-)$. This is slightly subtle since, as mentioned above, $W_a(D_4^-)$ does not act on the set $\mathrm{Hom}(\pi_1(S^*),G)/G$ of conjugacy classes of fundamental group representations—one needs to either use a covering such as $\{(\theta_i,m_{ij})\}$ (or a suitable finite intermediate cover) or a quotient (such as the coefficients of the Fricke relation and the m_{ij}, on which $W_a(D_4^-)$ acts trivially).

Remark 12. Usually [10, 15, 5] the $\mathcal{F}_2$ action on the monodromy data is described in terms of the three-string (Artin/planar) braid groups

$$1\to P_3\to B_3\to S_3\to 1$$

whose centres act trivially. Quotienting this sequence by the centres $Z(B_3)=Z(P_3)\cong\mathbb{Z}$ one obtains

$$1\to\mathcal{F}_2\to\mathrm{PSL}_2(\mathbb{Z})\to S_3\to 1 \tag{9}$$

so $\mathcal{F}_2$ is identified with the level-two subgroup $\Gamma(2)$. However to obtain the full symmetries of $\mathrm{P_{VI}}$ one needs to extend $\mathcal{F}_2$ by S_4, as in (8), and not just by S_3; this is why we used $\mathrm{M}_{0,4}$. This just corresponds to pulling back (9) along the natural map $S_4\to S_3$ with kernel K_4 the Klein four group. Indeed there is a map $\mathrm{M}_{0,4}\to\mathrm{PSL}_2(\mathbb{Z})$ with kernel K_4 (arising from the cross-ratio $(\mathbb{P}^1)^4\setminus\mathrm{diags}\to B$), and in fact $\mathrm{M}_{0,4}\cong\mathrm{PSL}_2(\mathbb{Z})\ltimes K_4$ (cf. [2] p.206).

5. Properties of the 52 solutions

In this section we will list some of the properties of the 52 icosahedral solutions to $\mathrm{P_{VI}}$. See tables 1 and 2. The columns of table 1 are defined as follows:
• The degree is the number of branches that the solution has. (Recall solutions branch at $t\in\{0,1,\infty\}$.)
• The genus is the genus of the algebraic curve on which the function $y(t)$ becomes single-valued. This is computed using the Riemann–Hurwitz formula from the permutation representation of the cover.
• The column labelled 'Walls' lists the number of affine F_4 hyperplanes that the parameters of the solutions lie on. Since the Okamoto transformations reflect in these hyperplanes, this number is an invariant.
• The A_5 type of the solution is defined as follows. Recall the icosahedral rotation group A_5 has precisely five conjugacy classes. We label the four non-trivial classes, the rotations by $1/2,1/3,1/5,2/5$ of a turn, by the letters a,b,c,d respectively. Thus given a four-tuple M_1,M_2,M_3,M_4 of elements of the binary icosahedral group Γ we are listing the set of conjugacy classes of their image in $A_5=\Gamma/\pm$. (If there are only three classes listed, that means that the fourth class is the trivial class.) This set is an invariant of the icosahedral solution although it does not determine the equivalence class of the solution (compare e.g. rows 12 and 40).

	Degree	Genus	Walls	A_5 Type	Alcove Point	n	Good?	Group (Size)	Partitions
1	1	0	1	$a\,b\,c$	31, 19, 11, 1	192	$\circ$	1	
2	1	0	1	$a\,b\,d$	37, 17, 13, 7	192	$\circ$	1	
3	1	0	1	$a\,c\,d$	33, 21, 9, 3	192	$\circ$	1	
4	1	0	1	$b\,c\,d$	28, 16, 8, 4	192	$\circ$	1	
5	1	0	2	$b^2\,c$	26, 14, 6, 6	96	$\circ$	1	
6	1	0	2	$b^2\,d$	38, 18, 18, 2	96	$\circ$	1	
7	1	0	2	$b\,c^2$	22, 10, 10, 2	96	$\circ$	1	
8	1	0	2	$b\,d^2$	34, 14, 10, 10	96	$\circ$	1	
9	1	0	3	c^3	18, 6, 6, 6	32	$\circ$	1	
10	1	0	3	d^3	42, 18, 18, 6	32	$\circ$	1	
11	2	0	2	$b^2\,c^2$	42, 18, 10, 10	96	$\times$	2	1, 2
12	2	0	2	$b^2\,d^2$	50, 10, 6, 6	96	$\times$	2	1, 2
13	2	0	2	$c^2\,d^2$	42, 18, 6, 6	96	$\times$	2	1, 2
14	3	0	1	$b\,c^2\,d$	40, 16, 8, 8	288	$\times$	S_3	3, 2
15	3	0	1	$b\,c\,d^2$	40, 8, 4, 4	288	$\times$	S_3	3, 2
16	4	0	2	$a\,c^3$	33, 9, 9, 9	128	$\circ$	A_4	3
17	4	0	2	$a\,d^3$	51, 3, 3, 3	128	$\circ$	A_4	3
18	4	0	2	$c^3\,d$	30, 6, 6, 6	128	$\circ$	A_4	3
19	4	0	2	$c\,d^3$	42, 6, 6, 6	128	$\circ$	A_4	3
20	5	0	1	$b^2\,c\,d$	44, 12, 12, 4	480	$\times$	S_5	2^2, $2\,3$
21	5	0	2	$c^2\,d^2$	36, 12, 0, 0	240	$\times$	S_5	3, $2\,3$
22	6	0	1	$b\,c^2\,d$	34, 10, 2, 2	576	$\circ$	S_6	5, $2\,3$
23	6	0	1	$b\,c\,d^2$	46, 14, 10, 2	576	$\circ$	S_6	5, $2\,3$
24	8	0	1	$a\,c^2\,d$	39, 15, 3, 3	768	$\times$	A_8	$3\,5$, $2^2\,3$
25	8	0	1	$a\,c\,d^2$	45, 9, 9, 3	768	$\times$	A_8	$3\,5$, $2^2\,3$
26	9	1	2	$b\,c^3$	28, 4, 4, 4	288	$\circ$	A_9	$3\,5$
27	9	1	2	$b\,d^3$	52, 8, 8, 4	288	$\circ$	A_9	$3\,5$
28	10	0	2	$a^2\,c\,d$	48, 12, 6, 6	480	$\times$	$2^7\,3\,5$	$2^2\,3^2$
29	10	0	2	$b^3\,c$	46, 14, 14, 6	320	$\circ$	A_{10}	$2^2\,5$
30	10	0	2	$b^3\,d$	42, 2, 2, 2	320	$\circ$	A_{10}	$2^2\,5$
31	10	0	3	c^4	24, 0, 0, 0	80	$\circ$	A_{10}	$3\,5$
32	10	0	3	d^4	48, 0, 0, 0	80	$\circ$	A_{10}	$3\,5$
33	12	0	0	$a\,b\,c\,d$	43, 11, 7, 1	2304	$\times$	A_{12}	$2^2\,3^2$, $2^2\,3\,5$
34	12	1	1	$a\,b\,c^2$	37, 13, 5, 5	1152	$\times$	A_{12}	$3^2\,5$, $2^2\,3\,5$
35	12	1	1	$a\,b\,d^2$	49, 5, 5, 1	1152	$\times$	A_{12}	$3^2\,5$, $2^2\,3\,5$
36	12	1	1	$b^2\,c\,d$	38, 6, 6, 2	1152	$\times$	$2^9\,3^2\,5$	$2^2\,3^2$, $2\,5^2$
37	15	1	2	$b^3\,c$	36, 4, 4, 4	480	$\times$	A_{15}	$2^2\,3^2\,5$
38	15	1	2	$b^3\,d$	48, 8, 8, 8	480	$\times$	A_{15}	$2^2\,3^2\,5$
39	15	1	2	$b^2\,c^2$	32, 8, 0, 0	720	$\times$	S_{15}	$3^2\,5$, $2\,3\,5^2$
40	15	1	2	$b^2\,d^2$	44, 4, 0, 0	720	$\times$	S_{15}	$3^2\,5$, $2\,3\,5^2$
41	18	1	3	b^4	40, 0, 0, 0	144	$\circ$	$2^{14}\,3^4\,5\,7$	$3^2\,5^2$
42	20	1	1	$a\,b^2\,c$	41, 9, 9, 1	1920	$\times$	A_{20}	$2^4\,3^2\,5$, $2^2\,3^2\,5^2$
43	20	1	1	$a\,b^2\,d$	47, 7, 3, 3	1920	$\times$	A_{20}	$2^4\,3^2\,5$, $2^2\,3^2\,5^2$
44	20	1	3	$a^2\,c^2$	42, 18, 0, 0	480	$\times$	$2^{17}\,3^4\,5^2\,7$	$3^2\,5^2$, $2^2\,3^2\,5^2$
45	20	1	3	$a^2\,d^2$	54, 6, 0, 0	480	$\times$	$2^{17}\,3^4\,5^2\,7$	$3^2\,5^2$, $2^2\,3^2\,5^2$
46	24	1	2	$a\,b^3$	45, 5, 5, 5	768	$\times$	$2^{20}\,3^5\,5^2\,7\,11$	$2^4\,3^2\,5^2$
47	30	2	2	$a^2\,b\,c$	46, 14, 4, 4	1440	$\times$	$2^{24}\,3^6\,5^3\,7^2\,11\,13$	$2^2\,3^2\,5^4$, $2^4\,3^4\,5^2$
48	30	2	2	$a^2\,b\,d$	52, 8, 2, 2	1440	$\times$	$2^{24}\,3^6\,5^3\,7^2\,11\,13$	$2^2\,3^2\,5^4$, $2^4\,3^4\,5^2$
49	36	3	3	$a^2\,b^2$	50, 10, 0, 0	864	$\times$	$2^{23}\,3^4\,5\,7$	$3^4\,5^4$, $2^2\,3^4\,5^4$
50	40	3	3	$a^3\,c$	51, 9, 9, 9	320	$\times$	$2^{25}\,3^4\,5^2\,7$	$2^4\,3^4\,5^4$
51	40	3	3	$a^3\,d$	57, 3, 3, 3	320	$\times$	$2^{25}\,3^4\,5^2\,7$	$2^4\,3^4\,5^4$
52	72	7	3	$a^3\,b$	55, 5, 5, 5	576	$\times$	$2^{32}\,3^4\,5\,7$	$2^4\,3^8\,5^8$

TABLE 1. Properties of the 52 icosahedral solutions.

• The alcove point is the value of (sixty times) the unique four-tuple of equivalent parameters $(\theta_1, \theta_2, \theta_3, \theta_4)$ which lies in the closure of the standard alcove (7). We scale by 60 simply to clear the denominators. This is the 'parameter equivalence class'.
• The value n in the next column is the number of 7-tuples $\mathbf{m} \in S$ corresponding to that equivalence class. Thus each n is divisible by the corresponding degree and the sum of all the n's is 26688.
• Let k be the degree of one of the solutions. We will say the solution is 'good' if it has a representative (amongst the n/k coming from S) for which Jimbo's formula (cf. [16] and [5] Theorem 4) may be applied to give the asymptotics at $t = 0$ on every branch. A cross ($\times$) means the solution has such a good representative and that we can in principle apply the procedure of [5] to find the solution explicitly. Even if a solution is not good ($\circ$) there may well be other ways to identify the solution (see below).
• The column 'Group (Size)' lists the nonlinear monodromy group of the solution or at least the size of this group; this is the group generated by the permutations of the branches of the solution curve as t goes around $0, 1, \infty$ and so naturally appears as a subgroup of the symmetric group on k letters. (In other words it is the monodromy group of the solution curve, expressed as a branched cover of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, or equivalently the Galois group of this cover over the base field $\mathbb{C}(t)$.)
• The final column 'Partitions' lists the set of conjugacy classes of the three generators of the nonlinear monodromy group. These are conjugacy classes in Sym_k, where k is the degree of the solution, and so are written as partitions of k (representing the cycle lengths of the permutations). If there are less than 3 partitions listed (separated by commas) the *last* partition is repeated. E.g. in row 20 the degree is 5 and the three partitions indicated are $1+2+2, 2+3, 2+3$ (repeating the last one). The set of these partitions is an invariant.

Remark 13. Observe that there are a number of consecutive rows of table 1 which look the same except for having a different alcove point and having c, d swapped in their A_5 type. We will refer to these as *sibling* solutions. They occur because A_5 has a non-trivial outer automorphism (from the extension $1 \to A_5 \to S_5 \to \mathbb{Z}/2 \to 1$, i.e. by conjugating an element of A_5 by an odd permutation) which swaps the conjugacy classes c, d and preserves the others. The action of this outer automorphism on the additive side, on the solutions themselves, remains mysterious.

Remark 14. Observe that the first ten entries of table 1 correspond to solutions with only one branch. Looking at the A_5 type we see that in each case one of the local (linear) monodromies is (projectively) trivial. Thus these ten correspond to the list of equivalence classes of *pairs* of generators of the icosahedral group, i.e. to hypergeometric equations with icosahedral monodromy, i.e. to the ten icosahedral entries on Schwarz's list [24]. (Replacing a by $1/2$, b by $1/3$ etc. gives the bijection with this part of Schwarz's list.) As solutions of $\mathrm{P_{VI}}$ these are all equivalent to a constant solution (and also equivalent to the solution $y = t$, with parameters as listed in table 2). Thus it is tempting to view Okamoto's $W_a(F_4)$-action as the natural extension (to linear systems of the form (1)) of the equivalence relation used by Schwarz. But this is not quite right as Okamoto's action does not preserve the linear monodromy group (cf. remark 10). Rather the $W_a(F_4)$ action is the natural analogue for the nonlinear $\mathrm{P_{VI}}$ equation of Schwarz's equivalence relation (and this was how Okamoto was thinking of it in [22]); it does indeed preserve algebraicity

of P_{VI} solutions. The extension of the further step taken by Klein, of obtaining the entries of Schwarz's list as pullbacks along a rational map from the 'basic Schwarz list', is also possible (indeed Klein proves this, at least for the corresponding Fuchsian equations). However to construct such rational pullbacks explicitly, before the system upstairs has been found explicitly, is a difficult problem (moreover the P_{VI} solution is equivalent to constructing a complete algebraic family of such covers as the four pole positions move). Such a procedure has been described (modulo the difficulty mentioned) independently by Doran [8] and Kitaev [18]. (Kitaev's paper [18] also contains some new explicit examples—i.e. not equivalent to solutions previously constructed by other means—see 2) of Remark 17 below.)

Remark 15. The next solutions ($\leqslant 4$ branches) are simple deformations of known solutions, i.e. the same solution just with different parameters, as follows.

The solutions with two branches (11, 12 and 13) are equivalent to the solution $y = \pm\sqrt{t}$. (One first observes that if $\theta_2 = \theta_3$ and $\theta_1 + \theta_4 = 1$ then this is indeed a solution. Then one checks each of solutions 11, 12, 13 has a representative with such parameters, as listed in table 2. Finally one uses Jimbo's formula to see that a leading term of these solutions matches that of $y = \pm\sqrt{t}$.)

Similarly the three-branch tetrahedral solution

$$y = \frac{(s-1)(s+2)}{s(s+1)}, \qquad t = \frac{(s-1)^2(s+2)}{(s+1)^2(s-2)} \tag{10}$$

on p.592 of [13] is actually a solution on the whole line $\theta_1/2 = \theta_2 = \theta_3, \theta_4 = \frac{2}{3}$, amongst other possibilities. (Note β should be $-2/9$, not $-1/18$ in op. cit.) Solutions 14 and 15 have representatives on this line (see table 2). Their leading terms given by Jimbo's formula (on the two branches where it may be applied) are $\pm i\sqrt{3}t^{1/2}$, which match the Puiseux expansion of (10), so (10) gives both icosahedral solutions 14 and 15. (Using the P_{VI} equation the leading terms determine the whole Puiseux expansion and thus the entire solution.)

Next the four-branch dihedral solution in section 6.1 of [12]:

$$y = \frac{s^2(s+2)}{s^2+s+1}, \qquad t = \frac{s^3(s+2)}{2s+1} \tag{11}$$

is a solution if $\theta_1 = \theta_2 = \theta_3, \theta_4 = 1/2$. As above this gives the icosahedral solutions 16 and 17, with the parameters indicated in table 2.

Finally the four-branch octahedral solution

$$y = \frac{(s-1)^2}{s(s-2)}, \qquad t = \frac{(s+1)(s-1)^3}{s^3(s-2)} \tag{12}$$

on p.588 of [13] is a solution if $\theta_1 = \theta_2 = \theta_3$ and either

$$\theta_4 = 1 - 3\theta_1 \quad \text{or} \quad \theta_4 = 1 + 3\theta_1.$$

(The implicit version of this in [13] should read: $3\,y^4 - (4\,t + 4)\,y^3 + 6\,ty^2 - t^2 = 0$.)

This gives icosahedral solution 18, with parameters as in table 2. Solution 19 is slightly more elusive and looks not to have a representative in either family. However it is *equivalent* to a member of the second family: Take the solution (12), with

$(\theta_1, \theta_2, \theta_3, \theta_4) = (2, 2, 2, 11)/5$. Then apply the sequence of Okamoto transformations

$$s_1(s_2s_0s_3s_4)^2s_2$$

in the notation of [21] (we act on the left, so we do the right-most s_2 first). This is the transformation which reduces θ_4 by 2 so yields an explicit solution with parameters $(2, 2, 2, 1)/5$, which may be parameterised as follows:

$$y = \frac{7 + 22\,s + 7\,s^2}{8\,(1 + s + s^2)\,s\,(s + 2)}, \qquad t = \frac{1 + 2\,s}{s^3\,(s + 2)}.$$

The corresponding Puiseux expansion at zero has a leading term $7\;2^{1/3}t^{2/3}/16$ which matches the leading term given by Jimbo's formula for entry 19 of table 2, and so this is icosahedral solution 19.

(We remark that there is thus a problem with the nomenclature for the expressions (10)-(12), they are as much *icosahedral* as they are *tetrahedral* etc. This is on top of the fact that Hitchin's octahedral solution (12) is equivalent to a solution found independently by Dubrovin [9] starting from the tetrahedral reflection group, and similarly for (10) and the octahedral reflection group, cf. [5] Remark 14.)

Remark 16. The three Dubrovin–Mazzocco icosahedral solutions [9, 10] are equivalent to the solutions on rows $31, 32$ (10 branch siblings) and 41 (18 branches, genus one). To prove they are equivalent one can show that the unipotent monodromy data used in [10] may be mapped by an Okamoto transformation to a triple of generators of Γ (cf. [5] Remark 14). (It is sufficient to use the affine D_4 group, which acts trivially on the quadratic functions m_{ij} of the monodromy data [14].) Alternatively a simpler but less direct way to see this is to observe that the icosahedral solutions $31, 32, 41$ here have equivalent parameters to those of [10]. Then appeal to the classification of [10] of all such finite branching solutions. Observe also that apart from this 18 branch solution, all icosahedral solutions with more than 10 branches are good, and so their Puiseux expansions at 0 may be computed using Jimbo's formula. Solution 41 appears in Theorem C of the introduction; Jimbo's formula yields the leading term on 16 of the 18 branches at zero and for the other two (where the solution does not in fact branch) we used Okamoto transformations to transfer the leading terms given by Dubrovin and Mazzocco [10] (in fact one needs the first 2 terms in the Taylor expansion in order for $\mathrm{P_{VI}}$ to determine the series uniquely, and to compute these terms one needs the first 3 terms of the corresponding branches of Dubrovin–Mazzocco's solution, but these are easily found from the given leading terms in [10]). The equivalent parameters used in [10] were $\theta_1 = \theta_2 = \theta_3 = 0, \theta_4 = -2/3$ and by using Okamoto transformations it is straightforward to convert the parameterisation of Theorem C into a solution for these parameters. The result is:

$$y = \frac{1}{2} + \frac{\left(\begin{matrix} 128\,s^{18} - 2496\,s^{17} + 19728\,s^{16} + 4605216\,s^{15} - 53030400\,s^{14} + 229874976\,s^{13} - 600089472\,s^{12} + \\ 968994816\,s^{11} - 823777848\,s^{10} - 88169600\,s^{9} + 1204313064\,s^{8} - 1658437668\,s^{7} + 1282505784\,s^{6} - \\ 632776452\,s^{5} + 199216125\,s^{4} - 36900918\,s^{3} + 3168636\,s^{2} + 134172\,s - 38416 \end{matrix}\right)}{6u\left(\begin{matrix} 5776\,s^{15} - 85440\,s^{14} + 482880\,s^{13} - 1490080\,s^{12} + 13986240\,s^{11} - 58604928\,s^{10} + \\ 133381480\,s^{9} - 186525360\,s^{8} + 162484560\,s^{7} - 80442380\,s^{6} + 11088528\,s^{5} + \\ 12426960\,s^{4} - 9203395\,s^{3} + 3037020\,s^{2} - 496860\,s + 33124 \end{matrix}\right)}$$

with t, u, s exactly as in Theorem C.

Remark 17. Even if a solution is not 'good' it may well be accessible:

1) As already discussed, the smaller solutions (1-4 branches) are simple deformations of known solutions.

2) Page 12 of A. Kitaev's paper [18] contains an explicit formula for the solution on row 26 of table 1, the smallest genus one solution. Presumably the sibling solution (row 27) can be obtained similarly; in any case we will obtain it with our methods in section 7. (Also [18] (3.3) p.24 corresponds to row 21.)

3) The Dubrovin–Mazzocco icosahedral solutions are not good in the sense of table 1, but were found by adapting Jimbo's formula to their situation. A different adaptation will be made at the end of section 7 to find the asymptotics of the outstanding solutions.

Remark 18. The three largest solutions, rows $50, 51$ (genus three, 40 branch siblings) and row 52 (genus seven, 72 branches), are related to Lamé equations (certain second order Fuchsian ordinary differential equations having four singularities on $\mathbb{P}^1$, and no apparent singularities). Namely there are Lamé equations having these (projective) monodromy representations, given explicitly in the paper [1] of Beukers and van der Waall. Converting their equations into Fuchsian systems will give initial conditions for these three P_{VI} solutions, although evolving P_{VI} to get a closed form for these solutions is somewhat daunting. (The leading terms in the Puiseux expansions given by Jimbo's formula seem to give more information however.) Note that the corresponding isomonodromic deformation will not be within the space of Lamé equations—the deformed equations will have an apparent singularity. Said differently, we may follow R. Fuchs and think of P_{VI} as controlling isomonodromic deformations of rank two Fuchsian equations (rather than systems), having four non-apparent singularities at $z = 0, t, 1, \infty$ and an apparent singularity at $z = y$ (cf. e.g. [21] for the formulae). If we choose t such that $y = 0, 1, t$ or ∞ then this equation will have only the four non-apparent singularities, so will be a Heun equation (with finite monodromy) and for the last three solutions this Heun equation will be of Lamé type. All Heun/Lamé equations with finite monodromy should arise in this way.

As an example let us list the leading term at zero of the asymptotic expansion of each branch of solution 52. (We consider the representative with $\theta_1 = \theta_2 = \theta_3 = 1/2, \theta_4 = 2/3$.) The leading terms are each of the form $c \times t^{1-\sigma}$ where the coefficients c are given by Jimbo's formula ([16], [5] Theorem 4). To express the coefficients as algebraic numbers we raise them (or their real/imaginary parts) to sufficiently high powers until they become rational and then look at the continued fraction expansions. From table 1, the 72 branches over zero are partitioned into 4 two-cycles, 8 three-cycles and 8 five-cycles. The values of σ and c for one branch of each cycle are as follows:

$$\sigma = \frac{1}{2} : \ \frac{1}{2}\left(\pm 3\sqrt{5} \pm 6\,i\right)^{\frac{1}{2}}, \quad \sigma = \frac{1}{3} : \ \pm\left(6\sqrt{3} \pm 2i\sqrt{5}\right)^{\frac{1}{3}}, \quad \sigma = \frac{2}{3} : \ \pm\frac{2}{3}\left(3\sqrt{3} \pm i\sqrt{5}\right)^{\frac{1}{3}},$$
$$\sigma = \frac{1}{5} : \ \pm 2\,i\,6^{3/5}, \quad \sigma = \frac{2}{5} : \ \pm\frac{9}{7}\,i\,6^{1/5}, \quad \sigma = \frac{3}{5} : \ \pm\frac{4}{13}\,i\,6^{4/5}, \quad \sigma = \frac{4}{5} : \ \pm\frac{12}{19}\,i\,6^{2/5}.$$

To obtain the other leading terms for each cycle, one just multiplies by all the k-th roots of unity, where k is the cycle length. (Here σ is the same for all branches of each cycle and the cycle length is equal to the denominator of σ.) It is still a challenge to write down the polynomial $F(t, y) = 0$, of degree 72 in y defining the solution curve and having these leading terms in its Puiseux expansions over $t = 0$. The curve itself is determined abstractly by its permutation representation as a cover of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. This can easily

be computed applying the operations ω_i^2 to the seven-tuples of monodromy data. The corresponding permutations of the 72 branches thus obtained, going around the loops w_1, w_2 of section 4, are, respectively:

$$(1\,2)(3\,4)(5\,6)(7\,8)(9\,10\,11)(12\,13\,14)(15\,16\,17)(18\,19\,20)(21\,22\,23)(24\,25\,26)(27\,28\,29)(30\,31\,32)$$
$$(33\ldots37)(38\ldots42)(43\ldots47)(48\ldots52)(53\ldots57)(58\ldots62)(63\ldots67)(68\ldots72)$$

$$(1\,18\,39\,49\,21)(2\,15\,33\,45\,24)(3\,22\,59\,68\,20)(4\,25\,56\,65\,17)(5\,27\,50\,38\,11)(6\,30\,46\,37\,14)(7\,9\,69\,58\,29)(8\,12\,66\,55\,32)$$
$$(10\,34\,64)(13\,40\,72)(16\,70\,42)(19\,67\,36)(23\,47\,54)(26\,51\,62)(28\,57\,44)(31\,60\,48)(35\,52)(41\,43)(53\,71)(61\,63)$$

One easily computes, using Riemann–Hurwitz, that this represents a genus 7 Belyi curve, and wonders whether this curve is remarkable for any other reasons.

To end this section we will list (in table 2) a representative seven-tuple $\mathbf{m}$ for each solution. (Recall that $\mathbf{m}$ determines the overall conjugacy class of the triple M_1, M_2, M_3.) Thus from this data one can for example easily compute the full permutation representation of each solution curve as a cover of the three-punctured sphere. (It is too cumbersome to write them all directly.) Rather than write the numbers $m_i = \mathrm{Tr}(M_i)$ and $m_{ij} = \mathrm{Tr}(M_iM_j)$ it is simpler to write the rational numbers θ_i, σ_{ij}, where

$$\mathrm{Tr}(M_i) = 2\cos(\pi\theta_i) \qquad \text{and} \qquad \mathrm{Tr}(M_iM_j) = 2\cos(\pi\sigma_{ij})$$

with $0 \leqslant \theta_i, \sigma_{ij} \leqslant 1$.

6. The generic icosahedral solution

Looking carefully at the 'Walls' column of table 1, the author was surprised to see there is a zero, on row 33. Namely there is a solution whose parameters are generic in that they lie on *none* of the affine F_4 hyperplanes. (This clearly implies they also lie on none of the reflecting hyperplanes of Okamoto's affine D_4 action.)

Being convinced there must be some mistake this solution was pursued further and in this section we will present the explicit solution: an algebraic solution with 12 branches and generic parameters; the largest genus zero icosahedral solution. Note that this solution is generic in another sense too: it has the largest value 2304 of n in table 1, so a randomly chosen triple of generators of the icosahedral group is most likely to lead to this solution. (This number $2304 = 12 \cdot 4! \cdot 2^3$ also shows that the group $\widetilde{\mathrm{M}}_{0,4}$ is the smallest possible extension of $\mathcal{F}_2$ yielding Theorem 1: namely the $\mathcal{F}_2$ orbits have size 12 so the 2304 points of S correspond to $4! \cdot 2^3$ solutions of this type, each with 12 branches, and so we deduce that the group $\widetilde{\mathrm{M}}_{0,4}/\mathcal{F}_2 \cong S_4 \ltimes \Sigma$ acts simply transitively on this set of solutions.)

Observe also, from the A_5 type, that the (projective) linear monodromy of the corresponding linear system has one generator in each of the four non-trivial conjugacy class of A_5. In this sense we are at the opposite extreme to the Dubrovin–Mazzocco solutions, which have all the A_5 conjugacy classes equal.

The solution was constructed as described in [5], using Jimbo's asymptotic formula ([16], [5] Theorem 4), noting that this solution has 'good' representatives. (The idea of using precise knowledge of the asymptotics to determine algebraic solutions of P_{VI} was also used in [10].) The solution curve thus arrived at is:

1	(1/2, 0, 1/3, 1/5)	(1/2, 1/3, 1/5)	27	(2/5, 2/5, 2/3, 2/5)	(4/5, 3/5, 3/5)
2	(1/2, 0, 1/3, 2/5)	(1/2, 1/3, 2/5)	28	(1/2, 1/2, 1/5, 3/5)	(1/2,1/2, 2/3)
3	(1/2, 0, 1/5, 2/5)	(1/2, 1/5, 2/5)	29	(1/3, 1/3, 1/3, 4/5)	(2/3, 3/5, 3/5)
4	(1/3, 0, 1/5, 2/5)	(1/3, 1/5, 2/5)	30	(1/3, 1/3, 1/3, 2/5)	(2/3, 1/5, 1/5)
5	(1/3, 0, 1/3, 1/5)	(1/3, 1/3, 1/5)	31	(4/5, 4/5, 4/5, 4/5)	(1/3, 1/5, 0)
6	(1/3, 0, 1/3, 3/5)	(1/3, 1/3, 3/5)	32	(3/5, 3/5, 3/5, 3/5)	(3/5, 3/5, 3/5)
7	(1/3, 0, 1/5, 1/5)	(1/3, 1/5, 1/5)	33	(2/5, 1/2, 1/3, 4/5)	(4/5, 2/3, 2/3)
8	(1/3, 0, 2/5, 2/5)	(1/3, 2/5, 2/5)	34	(1/5, 1/3, 1/5, 1/2)	(1/3, 2/5, 1/3)
9	(1/5, 0, 1/5, 1/5)	(1/5, 1/5, 1/5)	35	(2/5, 1/3, 2/5, 1/2)	(1/5, 1/5, 4/5)
10	(3/5, 0, 3/5, 3/5)	(3/5, 3/5, 3/5)	36	(1/3, 1/5, 1/3, 2/5)	(2/5, 2/5, 1/2)
11	(1/3, 1/5, 1/5, 2/3)	(1/2, 1/3, 1/2)	37	(1/3, 1/3, 1/3, 1/5)	(1/5, 1/3, 1/2)
12	(1/3, 2/5, 2/5, 2/3)	(1/2, 1/3, 1/2)	38	(1/3, 1/3, 1/3, 3/5)	(1/3, 1/3, 1/2)
13	(1/5, 2/5, 2/5, 4/5)	(1/2, 3/5, 1/2)	39	(1/3, 4/5, 1/3, 4/5)	(2/3, 3/5, 0)
14	(2/5, 1/5, 1/5, 2/3)	(1/2, 1/3, 1/2)	40	(3/5, 2/3, 3/5, 2/3)	(2/3, 1/5, 1/3)
15	(4/5, 2/5, 2/5, 2/3)	(1/2, 1/3, 1/2)	41	(1/3, 1/3, 1/3, 1/3)	(1/3, 1/3, 3/5)
16	(1/5, 1/5, 1/5, 1/2)	(1/3, 2/5, 1/3)	42	(1/3, 1/2, 1/3, 4/5)	(2/3, 4/5, 3/5)
17	(2/5, 2/5, 2/5, 1/2)	(1/3, 4/5, 1/3)	43	(1/3, 1/2, 1/3, 2/5)	(2/3, 1/2, 1/3)
18	(1/5, 1/5, 1/5, 2/5)	(1/3, 1/5, 1/3)	44	(1/2, 1/5, 1/2, 1/5)	(1/3, 2/3, 0)
19	(2/5, 2/5, 2/5, 1/5)	(1/3, 3/5, 1/3)	45	(1/2, 2/5, 1/2, 2/5)	(1/5, 4/5, 0)
20	(2/5, 1/3, 1/5, 2/3)	(2/3, 1/3, 1/2)	46	(1/3, 1/3, 1/3, 1/2)	(1/2, 1/3, 3/5)
21	(1/5, 1/5, 2/5, 2/5)	(1/3, 1/2, 1/3)	47	(1/2, 1/2, 1/3, 1/5)	(1/2, 1/2, 3/5)
22	(1/5, 1/5, 2/5, 1/3)	(2/5, 1/3, 1/3)	48	(1/2, 1/2, 1/3, 2/5)	(1/2, 4/5, 1/2)
23	(2/5, 2/5, 1/5, 2/3)	(4/5, 1/3, 1/3)	49	(1/3, 1/2, 1/3, 1/2)	(2/3, 3/5, 3/5)
24	(1/2, 2/5, 1/5, 4/5)	(2/3, 1/3, 1/2)	50	(1/2, 1/2, 1/2, 1/5)	(1/2, 2/5, 1/3)
25	(2/5, 2/5, 1/2, 4/5)	(3/5, 2/3, 1/2)	51	(1/2, 1/2, 1/2, 2/5)	(1/2, 2/3, 1/5)
26	(1/5, 1/5, 1/5, 2/3)	(1/5, 2/5, 1/5)	52	(1/2, 1/2, 1/2, 2/3)	(1/2, 1/5, 2/5)

TABLE 2. Representative seven-tuples $(\theta_1, \theta_2, \theta_3, \theta_4)$, $(\sigma_{12}, \sigma_{23}, \sigma_{13})$.

Generic icosahedral solution

$$\begin{gathered}
(15524784\,t^2 - 5373216\,t + 1350000)\,y^{12} - (128381760\,t^2 - 13366080\,t)\,y^{11} + \\
(5425704\,t^3 + 496677744\,t^2 - 30539160\,t)\,y^{10} - \\
(14929920\,t^4 + 41364000\,t^3 + 866759680\,t^2 - 2928160\,t)\,y^9 + \\
(107546535\,t^4 - 508275750\,t^3 + 747613335\,t^2 - 1837080\,t)\,y^8 - \\
(24385536\,t^5 - 285548724\,t^4 - 2437066824\,t^3 + 74927724\,t^2 + 944784\,t)\,y^7 + \\
(58212000\,t^5 - 2865570750\,t^4 - 4456260900\,t^3 + 17631810\,t^2)\,y^6 - \\
(49787136\,t^6 - 904003584\,t^5 - 7215732804\,t^4 - 2130570936\,t^3 - 12872196\,t^2)\,y^5 - \\
(413500320\,t^6 + 3724484160\,t^5 + 4839581265\,t^4 + 162430110\,t^3 + 3750705\,t^2)\,y^4 + \\
(3001304640\,t^6 + 74794560\,t^5 + 2710584000\,t^4 - 380946240\,t^3)\,y^3 - \\
(940800000\,t^7 + 977540640\,t^6 - 726801696\,t^5 + 939255264\,t^4 - 72013536\,t^3)\,y^2 + \\
(1176000000\,t^7 - 1481095680\,t^6 + 765158400\,t^5)\,y - \\
(1920800000\,t^8 - 7212800000\,t^7 + 10522980864\,t^6 - 6913299456\,t^5 + 1728324864\,t^4)
\end{gathered}$$

Implicit differentiation enables us to confirm that the function $y(t)$ defined by this polynomial is indeed a solution to the Painlevé VI equation, with $(\theta_1, \dots \theta_4) = (\frac{2}{5}, \frac{1}{2}, \frac{1}{3}, \frac{4}{5})$. Since this represents a genus zero curve one can use a computer to find a rational parameterisation. The author is very grateful to Mark van Hoeij for performing this task for the above curve and finding that it may be parameterised simply, as in Theorem B in the introduction.

7. More examples

By now it is not too much extra trouble to produce other solutions (since the procedure was sufficiently systematised in order to find a simple version of the generic solution). Here are the remaining good genus zero icosahedral solutions:

Solution 20, genus zero, 5 branches, $(\theta_1, \theta_2, \theta_3, \theta_4) = (2/5, 1/3, 1/5, 2/3)$:

$$y = \frac{2\,(s^2+s+7)\,(5\,s-2)}{s\,(s+5)\,(4\,s^2-5\,s+10)}, \qquad t = \frac{27\,(5\,s-2)^2}{(s+5)\,(4\,s^2-5\,s+10)^2}$$

Solution 24, genus zero, 8 branches, $(\theta_1, \theta_2, \theta_3, \theta_4) = (1/2, 2/5, 1/5, 4/5)$:

$$y = \frac{s\,(s+4)\,(3\,s^4-2\,s^3-2\,s^2+8\,s+8)}{8\,(s-1)\,(s^2+4)\,(s+1)^2}, \qquad t = \frac{s^5\,(s+4)^3}{4\,(s-1)\,(s^2+4)^2\,(s+1)^3}$$

Solution 25, genus zero, 8 branches, $(\theta_1, \theta_2, \theta_3, \theta_4) = (2/5, 2/5, 1/2, 4/5)$:

$$y = \frac{s^2\,(5\,s^3+2\,s^2-4\,s-8)\,(s+4)^2}{4\,(s+1)^2\,(s^2+4)\,(s-1)\,(s^2+3\,s+6)}, \qquad t = \frac{s^5\,(s+4)^3}{4\,(s-1)\,(s^2+4)^2\,(s+1)^3}$$

Solution 28, genus zero, 10 branches, $(\theta_1, \theta_2, \theta_3, \theta_4) = (1/2, 1/2, 1/5, 3/5)$:

$$y = \frac{(s^5+5\,s^4-20\,s^3+75\,s+75)\,(s^2-5)\,(s^2+5)}{(s+1)^2\,(s^2-4\,s+5)\,(s+5)\,(s^4+6\,s^2-75)}, \quad t = \frac{2\,(s^2+5)^3\,(s^2-5)^2}{(s+5)^3\,(s^2-4\,s+5)^2\,(s+1)^3}$$

The next solution we can find using Jimbo's formula is the generic solution (number 33) already displayed. Beyond that we pass onto the higher genus solutions. In principle we can still find these, although eventually one will have trouble computing all the symmetric functions of the Puiseux series of the solutions on the branches.

For example solutions 34 and 35 both become single valued on the elliptic curve:

$$u^2 = (3s+5)(8s^2-5s+5) \tag{13}$$

and, as functions on this curve, the solutions are given explicitly as:

Solution 34, genus one, 12 branches, $(\theta_1, \theta_2, \theta_3, \theta_4) = (1/5, 1/3, 1/5, 1/2)$:

$$y = \frac{1}{2} + \frac{(3\,s+5)\,(8\,s^4-10\,s^3+12\,s^2-13\,s+11)}{2\,\left(2\,s^3-15\,s+5\right)u}$$

$$t = t_{34} = \frac{1}{2} - \frac{(8\,s^6+20\,s^3-15\,s^2+66\,s-15)}{2\,\left(8\,s^2-5\,s+5\right)\,u}$$

Solution 35, genus one, 12 branches, $(\theta_1, \theta_2, \theta_3, \theta_4) = (2/5, 1/3, 2/5, 1/2)$:

$$y = \frac{1}{2} + \frac{(3\,s+5)\,(16\,s^5-8\,s^4+18\,s^3-8\,s^2+115\,s+3)}{2\,\left(26\,s^3+60\,s^2+15\,s+35\right)\,u}, \quad t = t_{34}$$

Next, solution 36 is given by the functions below on the curve $u^2 = 3(5s+1)(8s^2-9s+3)$:

Solution 36, genus one, 12 branches, $(\theta_1, \theta_2, \theta_3, \theta_4) = (1/3, 1/5, 1/3, 2/5)$:

$$y = \frac{1}{2} + \frac{140\,s^6+1029\,s^5-1023\,s^4+360\,s^3-288\,s^2+27\,s+27}{18\,u\,(s+1)\,(7\,s^3-3\,s^2-s+1)}$$

$$t = \frac{1}{2} + \frac{40\,s^6+540\,s^5-765\,s^4+540\,s^3-270\,s^2+27}{6\,u\,(8\,s^2-9\,s+3)\,(s+1)^2}$$

The next two solutions are related to the Valentiner group and will appear in section 8 and the outstanding good solutions with fewer than 20 branches are:

Solution 39, genus one, 15 branches, $(\theta_1, \theta_2, \theta_3, \theta_4) = (1/3, 4/5, 1/3, 4/5)$:

$$y = \frac{1}{2} + \frac{14\,s^5+61\,s^4-66\,s^3-660\,s^2-900\,s-225}{6\,\left(s+1\right)\left(s^2-5\right)u}$$

$$t = \frac{1}{2} - \frac{(2\,s^7+10\,s^6-90\,s^4-135\,s^3+297\,s^2+945\,s+675)\,u}{18\,\left(4\,s^2+15\,s+15\right)^2\left(s^2-5\right)}$$

where $u^2 = 3\,\left(s+5\right)\left(4\,s^2+15\,s+15\right)$.

Solution 40, genus one, 15 branches, $(\theta_1, \theta_2, \theta_3, \theta_4) = (3/5, 2/3, 3/5, 2/3)$:

$$y = \frac{1}{2} - \frac{2\,s^9+20\,s^8+53\,s^7-89\,s^6-605\,s^5-851\,s^4-1389\,s^3-5775\,s^2-10125\,s-5625}{2\,\left(s^2-5\right)\left(s^2-6\,s-15\right)\left(s^2+4\,s+5\right)u}$$

where t, u, s are as for solution 39 above.

Now we will fill in the gaps and explain how one may find the outstanding solutions for which Jimbo's formula cannot be applied directly, namely no.s $22, 23, 27, 29, 30$. Upon inspection one finds that these solutions always have a regular branch at zero (namely there is a cycle of lenth one in the permutation of the branches of the solution curve at zero). Thus we need to find the leading term in the Taylor/Laurent expansion of the solutions on the regular branches (the leading terms of the Puiseux expansions on the other branches still being given by Jimbo's formula). First we observe that each of these five solutions has a representative for which

$$\theta_1 + \theta_2 = \sigma$$

on one branch at zero (the regular branch), where $2\cos(\pi\sigma) = m_{12} = \mathrm{Tr}(M_1 M_2)$, $0 < \mathrm{Re}(\sigma) < 1$. Then the leading term is given by the following result.

Lemma 19. *If $\theta_1 + \theta_2 = \sigma$ on a branch of a solution to Painlevé VI with finite linear monodromy group, then the leading term of the Laurent expansion at zero of the solution is*

$$y(t) = \frac{\theta_1}{\theta_1+\theta_2} t + O(t^2)$$

Sketch. We proceed as in Jimbo's article [16]; As $t \to 0$ the system (1) degenerates into two hypergeometric systems ([16] (2.13, 2.14)) and the fundamental solutions and monodromy data can be related explicitly. Solving the Riemann–Hilbert problems for the two hypergeometric systems gives the asymptotics for the isomonodromic family of systems (1) (see [16] (2.15)), and therefore also for the $\mathrm{P_{VI}}$ solution.

In the case we are considering the condition $\theta_1+\theta_2=\sigma$ forces one of the hypergeometric systems ([16] (2.14)) to be reducible, and in fact abelian—since the monodromy group is finite. This makes the corresponding Riemann–Hilbert problem very easy to solve and yields the stated formula for the leading term. □

Remark 20. This can almost be guessed directly: substituting $y = a_1 t + a_2 t^2$ into $\mathrm{P_{VI}}$ gives the leading term

$$\frac{((\theta_2+\theta_1)a_1-\theta_1)((\theta_2-\theta_1)a_1+\theta_1)}{2a_1(1-a_1)}t^{-1}$$

at zero. Thus if $\theta_2=\theta_1$ then the value of a_1 in the lemma is forced, and moreover our five examples all have representatives with $\theta_2=\theta_1$.

The explicit formulae for these five solutions thus obtained are as follows.

Solution 22, genus zero, 6 branches, $(\theta_1,\theta_2,\theta_3,\theta_4)=(1/5,1/5,2/5,1/3)$:

$$y = \frac{-54\,s\,(s-7)}{(s^4-20\,s^2-35)\,(s+1)\,(s-4)}, \qquad t = \frac{432\,s}{(s+5)\,(s-4)^2\,(s+1)^3}$$

Solution 23, genus zero, 6 branches, $(\theta_1,\theta_2,\theta_3,\theta_4)=(2/5,2/5,1/5,2/3)$:

$$y = \frac{18\,s\,(s-3)}{(s-4)\,(s+1)\,(s^2+5)}, \qquad t = \frac{432\,s}{(s+5)\,(s-4)^2\,(s+1)^3}$$

Solution 27, genus one, 9 branches, $(\theta_1,\theta_2,\theta_3,\theta_4)=(2/5,2/5,2/3,2/5)$:

$$y = \frac{1}{2}+\frac{(350\,s^3+63\,s^2-6\,s-2)}{30\,(2\,s+1)\,su} \qquad t = \frac{1}{2}+\frac{(25\,s^4+170\,s^3+42\,s^2+8\,s-2)\,u}{54\,(5\,s+4)^2\,s^3}$$

where u and s satisfy $u^2 = s\,(8\,s+1)\,(5\,s+4)$.

Solution 29, genus zero, 10 branches, $(\theta_1,\theta_2,\theta_3,\theta_4)=(1/3,1/3,1/3,4/5)$:

$$y = \frac{(s+2)\,(s^2+1)\,(2\,s^2+3\,s+3)\,s^2}{2\,(s^2+s+1)\,(3\,s^2+3\,s+2)}, \qquad t = t_{29} = \frac{(s+2)\,(2\,s^2+3\,s+3)^2\,s^5}{(2\,s+1)\,(3\,s^2+3\,s+2)^2}$$

Solution 30, genus zero, 10 branches, $(\theta_1,\theta_2,\theta_3,\theta_4)=(1/3,1/3,1/3,2/5)$:

$$y = \frac{(s+2)\,(2\,s^2+3\,s+3)\,(7\,s^2+10\,s+7)\,s^4}{(3\,s^2+3\,s+2)\,(4\,s^6+12\,s^5+15\,s^4+10\,s^3+15\,s^2+12\,s+4)}, \quad t = t_{29}$$

Remark 21. (Added May 2005.) The remaining solutions (except the Valentiner solutions, which will appear in section 8 below, and solutions 42 and 43) may be obtained from known solutions using the quadratic transformations defined in 1991 by Kitaev [19]. The basic idea is as follows. Given an icosahedral Fuchsian system A with A_5 type $a^2\xi\eta$ for some $\xi, \eta \in \{a, b, c, d\}$ (i.e. with two local monodromies, say at 0 and ∞, of order two in $\mathrm{PSL}_2(\mathbb{C})$) we can pull back along the map $w \mapsto z = w^2$, and remove the resulting apparent singularities, to get a Fuchsian system B with A_5 type $\xi^2\eta^2$. Isomonodromic deformations of A correspond to isomonodromic deformations of B, and one can obtain formulae relating the corresponding $\mathrm{P_{VI}}$ solutions. In practice the formulae are much simpler at different (Okamoto equivalent) values of the parameters (see [23] (2.7) and also the recent article [26]). In the cases at hand, this procedure gives an algebraic relation with a solution having half the number of branches; Examining Table 1 we see solution 31 $\Rightarrow$ solution 44 and in turn solution 44 $\Rightarrow$ solution 50. Similarly

$$32 \Rightarrow 45 \Rightarrow 51, \qquad 39 \Rightarrow 47, \qquad 40 \Rightarrow 48, \qquad 41 \Rightarrow 49 \Rightarrow 52.$$

(Some work is still required to obtain efficient parameterisations of the solutions obtained in this way.) Fortunately the only possible simplification to the rest of this article is 21 $\Rightarrow$ 28, which was no trouble anyway.

In any case this gave us motivation to construct the remaining solutions 42 and 43 using our original method, essentially completing the construction of all icosahedral solutions:

Solution 42, genus one, 20 branches, $(\theta_1, \theta_2, \theta_3, \theta_4) = (1/3, 1/2, 1/3, 4/5)$:

$$y = \frac{1}{2} + \frac{(8\,s^6 - 28\,s^5 + 85\,s^4 - 196\,s^3 + 214\,s^2 - 196\,s + 41)\,(s+3)}{6\,(s^2+1)\,(3\,s^2 - 4\,s + 5)\,u}$$

$$t = \frac{1}{2} - \frac{(s+3)\,\mathrm{P}}{2\,(s^2+1)^2\,u^3}$$

where $u^2 = 3\,(s+3)\,(8\,s^2 - 13\,s + 17)$, and

$$\mathrm{P} = 8\,s^{10} + 100\,s^7 - 135\,s^6 + 834\,s^5 - 1205\,s^4 + 2280\,s^3 - 1365\,s^2 + 890\,s + 321.$$

Solution 43, genus one, 20 branches, $(\theta_1, \theta_2, \theta_3, \theta_4) = (1/3, 1/2, 1/3, 2/5)$:

$$y = \frac{1}{2} + \frac{(s+3)\,\mathrm{Q}}{18\,(s^2+1)\,(s^6 - 7\,s^4 + 42\,s^3 - 45\,s^2 + 34\,s + 7)\,u}$$

where

$$\mathrm{Q} = 28\,s^9 - 235\,s^8 + 556\,s^7 - 1334\,s^6 + 2174\,s^5 - 3854\,s^4 + 4360\,s^3 - 4738\,s^2 + 2362\,s - 1047$$

and t, u, s are as for solution 42.

8. The Valentiner Solutions

The Valentiner reflection group is the subgroup of $\mathrm{GL}_3(\mathbb{C})$ generated by the complex reflections (cf. e.g. [25]):

$$r_1 = \begin{pmatrix} 0 & -\omega^2 & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad r_2 = -\frac{1}{2}\begin{pmatrix} -1 & \omega\tau & \frac{\omega^2}{\tau} \\ \frac{\tau}{\omega} & \tau^{-1} & \omega \\ \frac{\omega}{\tau} & \omega^2 & -\tau \end{pmatrix} \quad r_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $\omega = \exp(2\pi i/3)$, $\tau = (1+\sqrt{5})/2$. It has order 2160 and the corresponding projective group in $\mathrm{PGL}_3(\mathbb{C})$ is isomorphic to A_6 the alternating group on six letters.

We wish to apply the procedure of [5] section 2 to this triple of generating reflections to obtain a triple of elements of $\mathrm{SL}_2(\mathbb{C})$. By definition the corresponding $\mathrm{SL}_2(\mathbb{C})$ triple (M_1, M_2, M_3) has invariants

$$\mathrm{Tr}(M_1) = \frac{t_1}{n_1} + \frac{n_1}{t_1}, \qquad \mathrm{Tr}(M_2) = \frac{t_2}{n_1} + \frac{n_1}{t_2}, \qquad \mathrm{Tr}(M_3) = \frac{t_3}{n_1} + \frac{n_1}{t_3},$$
$$\mathrm{Tr}(M_1M_2) = \frac{t_{12}}{t_1t_2}, \qquad \mathrm{Tr}(M_2M_3) = \frac{t_{23}}{t_2t_3}, \qquad \mathrm{Tr}(M_1M_3) = \frac{t_{13}}{t_1t_3},$$
$$\mathrm{Tr}(M_4) = \mathrm{Tr}(M_3M_2M_1) = \frac{n_2}{n_3} + \frac{n_3}{n_2}$$

where $t_{jk} = \mathrm{Tr}(r_jr_k) - 1$, t_j is a choice of square root of $\det(r_j)$, and the n_j are chosen square roots of the eigenvalues of the product $r_3r_2r_1$ (which we are thus choosing an order of too). Here each of the reflections r_j is of order two so we take may take the invariant $t_j = i$ for each j. Next, the product $r_3r_2r_1$ has eigenvalues $\{\exp(2\pi i\frac{5}{30}), \exp(2\pi i\frac{11}{30}), \exp(2\pi i\frac{29}{30})\}$, so we may take

$$n_1 = \exp(5\pi i/30), \quad n_2 = \exp(11\pi i/30), \quad n_3 = \exp(29\pi i/30).$$

Also we compute:

$$\mathrm{Tr}(r_1r_2) = 0, \qquad \mathrm{Tr}(r_2r_3) = 0, \qquad \mathrm{Tr}(r_1r_3) = 1.$$

Then the corresponding $\mathrm{SL}_2(\mathbb{C})$ invariants are:

$$m_1 = m_2 = m_3 = 1, \quad m_4 = 2\cos(3\pi/5), \quad m_{12} = m_{23} = 1, \quad m_{13} = 0.$$

Thus the θ parameters are $(1/3, 1/3, 1/3, 3/5)$, since $2\cos(\pi/3) = 1$, and one finds then that the corresponding $\mathrm{SL}_2(\mathbb{C})$ triple generates the binary icosahedral group and corresponds to row 38 of tables $1, 2$.

In particular if we are able to find the corresponding $\mathrm{P_{VI}}$ solution then (as was done in [5] for the Klein group) we can explicitly construct an isomonodromic family of rank three Fuchsian equations (with four poles on $\mathbb{P}^1$) having monodromy group equal to the Valentiner reflection group, generated by reflections.

Rather than repeat the details (which are exactly as in [5]) we just give the $\mathrm{P_{VI}}$ solution:

Solution 38, genus one, 15 branches, $(\theta_1,\theta_2,\theta_3,\theta_4)=(1/3,1/3,1/3,3/5)$:

$$y=\frac{1}{2}-\frac{250\,s^6+500\,s^5+518\,s^4+261\,s^3+76\,s^2+13\,s+2}{2\,(s+2)\,(5\,s+1)\,(5\,s^3+6\,s^2+3\,s+1)\,u}$$

$$t=\frac{1}{2}-\frac{3\,(500\,s^7+925\,s^6+1164\,s^5+830\,s^4+340\,s^3+105\,s^2+20\,s+4)}{2\,(s+2)^2\,(5\,s+1)\,u^3}$$

where (u,s) lies on the elliptic curve

$$u^2=\left(4\,s^2+s+1\right)(5\,s+1)\,.$$

However, unlike in the case of the Klein reflection group, the Valentiner group has three inequivalent triples of generating reflections (above we used the standard generating triple whose product has eigenvalues involving the exponents of the group). This is similar to the case of the icosahedral reflection group studied in [10], which also has three inequivalent triples of generating reflections (leading to the icosahedral solutions on rows $31,32,41$) although now all three solutions are elliptic and the largest has 24 branches.

The second generating triple gives the sibling solution of that above and arises by replacing r_1 above by

$$r_1=\begin{pmatrix}0&0&-\omega\\0&1&0\\-\omega^2&0&0\end{pmatrix}.$$

Then the product $r_3r_2r_1$ has eigenvalues $\exp(2\pi i\frac{5}{30})$, $\exp(2\pi i\frac{17}{30})$, $\exp(2\pi i\frac{23}{30})$, and similarly to above one finds the correpsonding $\mathrm{SL}_2(\mathbb{C})$ triple corresponds to row 37 of tables 1 and 2. The corresponding $\mathrm{P_{VI}}$ solution is:

Solution 37, genus one, 15 branches, $(\theta_1,\theta_2,\theta_3,\theta_4)=(1/3,1/3,1/3,1/5)$:

$$y=\frac{1}{2}-\frac{1000\,s^8+2425\,s^7+4171\,s^6+3805\,s^5+1999\,s^4+874\,s^3+244\,s^2+58\,s+4}{4\,(s+2)\,(25\,s^6+135\,s^5+111\,s^4+91\,s^3+36\,s^2+6\,s+1)\,u}$$

with t,u,s as for solution 38 above.

Finally the third generating triple of the Valentiner reflection group arises by replacing r_1 above by the reflection

$$r_1=\frac{1}{2}\begin{pmatrix}1-\tau&\tau&1\\\tau&1&1-\tau\\1&1-\tau&\tau\end{pmatrix}.$$

Then the product $r_3r_2r_1$ has eigenvalues $\exp(2\pi i\frac{2}{12})$, $\exp(2\pi i\frac{5}{12})$, $\exp(2\pi i\frac{11}{12})$, and similarly to above one now finds the correpsonding $\mathrm{SL}_2(\mathbb{C})$ triple corresponds to row 46 of tables 1 and 2. The corresponding $\mathrm{P_{VI}}$ solution has 24 branches so is larger than any previously constructed solution (and currently there are no known elliptic solutions of higher degree, regardless of whether they have been explicitly constructed). For this solution the previous method of constructing the solution polynomial, involving computing the symmetric functions of the Puiseux expansion at 0 of the solution branches, no longer works. (For example in the worst case one faces a sum of $\binom{24}{12}$ terms, each of which is a 12-fold product of Puiseux expansions with many terms.) Instead one can obtain some coefficients in this

way and then use the expected Okamoto symmetries of the solution to determine the outstanding coefficients, by solving some sparse overdetermined linear equations. (One then checks by implicit differentiation that the resulting polynomial indeed defines a $\mathrm{P_{VI}}$ solution.) Using Mark van Hoeij's algorithms (in the Maple algebraic curves package) we then obtain the following parameterisation:

Solution 46, genus one, 24 branches, $(\theta_1,\theta_2,\theta_3,\theta_4)=(1/3,1/3,1/3,1/2)$

$$y=\frac{1}{2}-\frac{\mathrm{P}}{2\,(3\,s^2-2\,s+2)\,\mathrm{R}\,u},\qquad t=\frac{1}{2}+\frac{(s^2+4\,s-2)\,\mathrm{Q}}{2\,(s+2)\,(3\,s^2-2\,s+2)^2\,u^3}$$

where

$$\mathrm{P}=16\,s^{11}+72\,s^{10}+50\,s^9-242\,s^8-3143\,s^7+6562\,s^6-8312\,s^5+9760\,s^4-9836\,s^3+6216\,s^2-2288\,s+416,$$

$$\mathrm{Q}=8\,s^{10}+16\,s^9+24\,s^8-84\,s^7+429\,s^6-312\,s^5+258\,s^4-288\,s^3+288\,s^2-128\,s+32,$$

$$\mathrm{R}=26\,s^6+18\,s^5-75\,s^4+50\,s^3+270\,s^2-312\,s+104,$$

and where (u,s) lies on the elliptic curve

$$u^2=\left(8\,s^2-7\,s+2\right)(s+2)\,.$$

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